Unusual period doublings in the linear-logistic map

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In the route to chaos, the linear-logistic map $g(x)$, consisting of a left linearly increasing portion joined to a right logistic portion at its maximum, has an unusual behavior. At every other period-doubling bifurcation point, the "slope" of $g^m(x)$ at any of the m stable fixed points is 0 instead of -1 for the usual period-doubling route. This map possesses rather unusual features in the Lyapunov exponent versus parameter graphs, the $g^m(x)$ graphs, the values of the scale reduction factor, and that of the superstable parameters. These properties suggest that either every other period doubling is different from the Feigenbaum type, or that all the period doublings are usual but with changes occurring extremely rapidly at every other one.

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I. INTRODUCTION

In one-dimensional continuous maps with negative Schwarzian derivatives, the route to chaos is through the Feigenbaum period-doubling bifurcations. If r_k denotes the parameter r of the map at which bifurcation from the 2^k- to the 2^{k+1}-cycle occurs and

$$\delta_k = (r_k - r_{k-1})/(r_{k+1} - r_k) , \tag{1}$$

which is the usual scale reduction factor, it is found that, for all such maps, δ, defined as the limit of δ_k as $k \to \infty$, has a universal value of 4.6692 [1].

Here we present some interesting results obtained in a numerical study of an asymmetric map which has been observed experimentally. It is a special case of the map discovered in experiments with a forced nonlinear oscillator: the map can be used for modeling the observed voltage across a nonlinear diode in series with an inductor, a resistor, and a sinusoidal voltage source [2]. Further, it has been regarded as a model of an impact oscillator [3]. It is described by the equation

$$g(x_n) = \begin{cases} 2rx_n & \text{if } 0 \le x_n < \frac{1}{2} \\ 4rx_n(1-x_n) & \text{if } \frac{1}{2} \le x_n \le 1 \end{cases} . \tag{2}$$

We shall call $g(x_n)$ the *linear-logistic map*, as it consists of a linear part and the logistic portion

$$h(x_n) = 4rx_n(1-x_n) . \tag{3}$$

For the above asymmetric map, the slope is not well defined everywhere because at the extremum, the left-hand derivative is not equal to the right-hand derivative. We shall take the slope at that point to be equal to that derivative with the smaller magnitude.

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II. PERIOD DOUBLING

A. Bifurcation diagram

Figure 1 shows the bifurcation diagram of the linear-logistic map $g(x_n)$ undergoing a series of period doublings. For $r \le r_{2,ss} = (1+\sqrt{5})/4 = 0.809016994$, where $r_{2,ss}$ is the parameter value at the superstable 2-cycle, the portion of the diagram resembles that for the logistic map. However when $r \ge r_{2,ss}$, they are different.

B. Period doublings in logistic map

Between two successive period doublings in the periodic region of the logistic map $h(x_n)$, the following two processes are observed to occur.

(a) At the birth of a 2^k-cycle, the stability-determining slope of the curve $h^{2^k}(x_n)$ at the 2^k fixed points is $+1$, which thereafter gradually decreases over a finite range Δr_a toward 0 when the cycle becomes superstable.

(b) From this superstability condition, the slope decreases gradually over a finite range Δr_b toward -1, at which the cycle becomes unstable and it period doubles into a 2^{k+1}-cycle.

C. Period doublings in linear-logistic map

There are also period doublings in the linear-logistic map $g(x_n)$ which, however, seem to occur whenever the stability-determining slope is alternately -1 and 0, quite unlike that in the logistic map when period doublings always occur whenever the slope is -1. More precisely, in the linear-logistic map we observe the following.

At the birth of a 2^k-cycle by the usual period-doubling bifurcation, the slope of g^{2^k} at each of its 2^k-cycle elements is $+1$, where k is 1,3,5,7,9,. . . . With increasing r, this slope decreases steadily till it is 0 when the 2^k-cycle is superstable. Surprisingly, an infinitesimally small increase in r induces an unusual period doubling into a 2^{k+1}-cycle. This 2^{k+1}-cycle is superstable at its creation.

With increasing r, the slope of the function $g^{2^{(k+1)}}$ at any of its 2^{k+1} elements steadily decreases in a manner

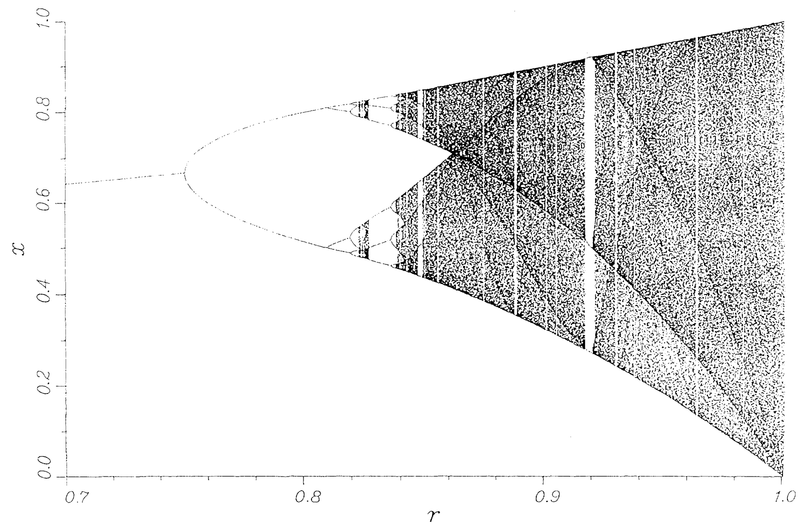

FIG. 1. Bifurcation diagram of the linear-logistic map.

similar to that in the logistic map, until it attains a value of -1 when there is the usual period doubling into a 2^{k+2}-cycle: at this value of r, the slope of $g^{2^{(k+2)}}$ at each of its 2^{k+2}-cycle elements is $+1$.

Subsequently, with increasing r, the process described above between the births of the 2^k- and 2^{k+2}-cycles is repeated. Thus period doubling occurs whenever the slope of g^{2^k} at each of its elements is 0 or -1, depending on whether k is odd or even, respectively.

III. UNUSUAL FEATURES OF BIFURCATIONS

Evidence of other unusual features of the bifurcations in the linear-logistic map is given here.

A. Lyapunov exponent

In the periodic region, when the Lyapunov exponent λ of the linear-logistic map attains a local minimum at $r_{n,\mathrm{ss}}$ (the value of r at the superstable n-cycle, where $n=2^{2k+1}$ and $k=0,1,2,\ldots$), a period doubling *occurs* in contrast to that in the logistic map. Besides these unusual period doublings at the superstable cycles, the usual period doublings also occur when $\lambda=0$.

Hence as r increases in the linear-logistic map, we have the normal type of period doubling which is followed by the unusual type, and so on. Thus the transition of orbits with period 2^{2k} to period 2^{2k+1} is normal, while it is unusual for those with period 2^{2k+1} to period 2^{2k+2}.

B. Slope of g^2

We observe that for a very small increment in r of 10^{-32}, a period doubling occurs immediately beyond the "superstable" $r_{2,\mathrm{ss}}$ value, a process which is not encountered in the logistic map. Note that at $r_{2,\mathrm{ss}}$, the slope of g^2 is not well defined at each cycle element as there is a discontinuity: it is zero on one side and negative on the other. With an infinitesimally small increase of r (i.e., less than 10^{-N_D}, where N_D is the number of decimal places used in the computation), a very small switchback appears, giving rise to a period doubling from a 2- to a 4-cycle. Thus at this unusual superstable point, an unusual period doubling has occurred in contrast to that in the logistic map.

When r lies between $r_{2,\mathrm{ss}}$ and 0.85, say, the slope of g^2 at each intersection with the $y=x$ line is negative, with a magnitude larger than 1, so that within this parameter range there are no stable fixed points of g^2. Hence a stable 2-cycle does not exist beyond $r_{2,\mathrm{ss}}$, thereby confirming the abrupt bifurcation of the superstable 2-cycle into a superstable 4-cycle. Similar arguments also apply to higher-order cycles.

C. Values of the scale reduction factor δ_k

The values of δ_k defined in Eq. (1) for this map, together with those reported for other exponent-asymmetry maps [4,5], are shown in Table I. While all these values exhibit oscillatory divergent behavior, those for the linear-logistic map show the most divergent behavior with the values of δ_k, where k is an even number larger

TABLE I. Values of δ_k, defined in Eq. (1), for the linear-logistic map are given in column 2. These values show the most divergent behavior as compared to those of maps in Refs. [4] and [5], given in columns 3 and 4, respectively.

k	δ_k		
	Linear-logistic map	Map in Ref. [4]	Map in Ref. [5]
1	5.921	3.643	
2	5.775	6.625	6.119
3	3.471	7.588	6.981
4	3.367×10^2	6.548	7.027
5	2.991	18.010	7.633
6	6.796×10^5	6.202	7.076
7	2.936		8.892
8	3.366×10^{12}		7.128
9	1.555		10.767

TABLE II. The first seven superstable parameters in the period-doubling sequence of the linear-logistic map.

Period n	$r_{n,\mathrm{ss}}$
1	0.500 000 000 000 000 000 000 000 000 000 00
2	0.809 016 994 374 947 424 102 293 417 182 82
4	0.809 016 994 374 947 424 102 293 417 182 83
8	0.822 180 394 705 424 074 465 217 207 469 22
16	0.822 180 394 705 424 074 465 217 207 469 23
32	0.822 192 062 630 828 624 785 788 402 301 04
64	0.822 192 062 630 828 624 785 788 402 301 05

than 2, being very huge, and it increases faster than that of the geometric progression. This unusually large increase of every other value of δ_k must be associated with the very unusual period doublings which occur alternately with the usual type.

D. Values of $r_{n,\mathrm{ss}}$

By direct computation, the values of the superstable parameters $r_{n,\mathrm{ss}}$ are given in Table II, which shows that, to 31 decimal places, $r_{2,\mathrm{ss}}=r_{4,\mathrm{ss}}$, and $r_{8,\mathrm{ss}}=r_{16,\mathrm{ss}}$, i.e., $r_{2^{2m+1},\mathrm{ss}}=r_{2^{2m+2},\mathrm{ss}}$ for integral values of m. From this, we can conclude that every other combined process, consisting of process (b) followed by process (a) defined in Sec. II B, occupies a very short unobservable range of Δr, which therefore gives rise to the impression that every other period doubling occurs at the superstable parameter.

E. Symbolic dynamics

We have used the *word-lifting technique* of symbolic dynamics to determine the parameters of the two superstable kneading sequences RC and RLRC [6]: these two values are identical, and coincide with that given above for $r_{2,\mathrm{ss}}$, verifying that there is indeed a very unusually abrupt bifurcation of the superstable 2-cycle into a superstable 4-cycle.

F. Bifurcation diagrams

In the bifurcation diagram for the linear-logistic map shown in Fig. 1, we see that at the birth of the unusual period doubling from periods 2–4, when $r=r_{2,\mathrm{ss}}$, each arm of the period-2 fork appears to continue on smoothly and remains stable after sprouting a stable offspring. Similarly, at the birth of the unusual period doubling from periods 8–16, when $r=r_{8,\mathrm{ss}}$, each arm of the period-8 fork appears to continue on smoothly and remains stable after sprouting an offspring. This process for the uppermost arm is shown in Fig. 2. This is to be contrasted with the usual period doubling in the logistic map where each arm splits into a two-prong pitchfork, after which the original period-2 arms become unstable. Note that at the other normal period-doubling points, namely from periods 1 to 2 and from periods 4 to 8, as shown in Fig. 1, this unusual offspring-creating process does not occur.

We can account for the above offspring-creating process by examining the g^4 map around the superstable parameter $r_{4,\mathrm{ss}}=0.809$. Near each of the two cycle elements, the slope of g^4 is negative on one side and zero on the other. With a very small increase of r, a very small switchback occurs near each of the original cycle elements. This tiny feature is shown in Fig. 3, which is a greatly enlarged region around one of the cycle elements with $r=0.81$. From the shapes and sizes of these switch-

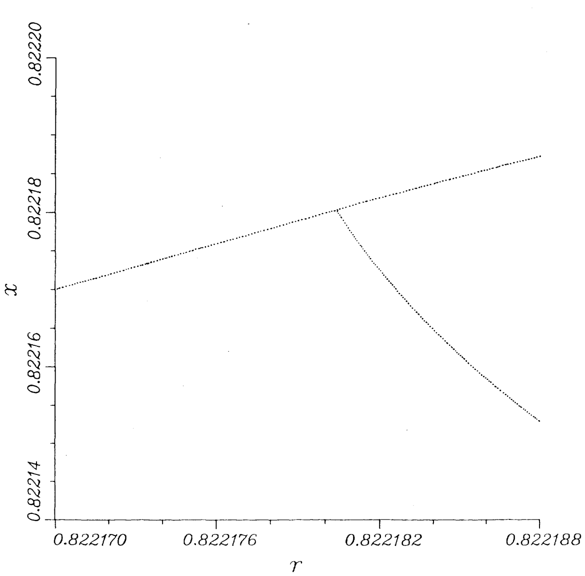

FIG. 2. A greatly enlarged bifurcation diagram showing the uppermost arm occurring in the unusual period doubling of the 8-cycle into the 16-cycle.

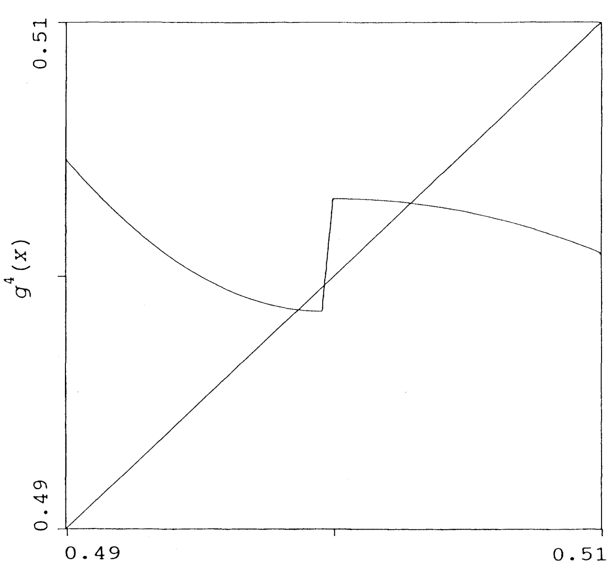

FIG. 3. Enlargement of a small portion of the graph of g^4 with $r=0.81$, which is the parameter at which a 4-cycle appears. Note that the unstable fixed point is close to a new stable cycle element.

backs, it is clear that each of the original cycle elements has become unstable and is very close to one of the new stable cycle elements generated. Thus, just after the superstable parameter, it appears that each arm of the period-2 fork continues on smoothly and remains stable after sprouting a stable offspring, when in fact it has become unstable but lies very close to one of the newborn stable cycle elements.

IV. INTERPRETATIONS

From the unusual features of the linear-logistic map given above, we may conclude either that there is a new mechanism of period doubling occurring at the superstable parameters, or that the mechanism is not new, but identical to that of the standard type except that it is slightly unusual. We shall now explain how this alternative explanation is possible.

According to standard theory, between two successive superstable cycles of the linear-logistic map, the combined process, consisting of processes (b) and (a) described above for the logistic map (with the function g replacing h), always occurs. Here for the linear-logistic map, we can regard every other combined process to occupy a very short, unobservable range of Δr (which is equal to the sum of Δr_a and Δr_b). Then as every other range Δr is unobservably small, every other period doubling will now appear to occur when the stability-determining slope is 0, whereas, when the range Δr is finite, period doubling is seen to occur normally, i.e., when the stability-determining slope is -1.

Hence the variation of the stability-determining slope as a function of r is a cycle with the following two components.

Component (i): For $k=1,3,5,7,9,\ldots$, as r is increased, process (a) is first observed (with the function g replacing h). However, when r is further increased by an infinitesimal amount, the superstable 2^k-cycle immediately becomes unstable. We can, in this interpretation, say that over an unobservable range Δr_b, the stability-determining slope decreases very quickly from 0 to -1 when period doubling occurs, thereby giving rise to a 2^{k+1}-cycle and the impression that period doubling occurs whenever the stability-determining slope is 0.

Component (ii): For $k=2,4,6,8,10,\ldots$, after a 2^k-cycle is born, the slope decreases very rapidly over an unobservable range Δr_a to 0 when the cycle becomes superstable. Subsequently, a process identical to process (b) of the logistic map is observed: the slope changes from 0 to -1 over a finite range Δr_b, at the end of which another period doubling occurs.

V. CONCLUSIONS AND SUMMARY

In the linear-logistic map, period doublings are observed to occur whenever the stability-determining slope at the fixed point is either -1 or 0. These period doublings observed at the superstable points have not been previously reported, to our knowledge. With the other unusual features given above, they suggest either that there is a new mechanism of period doubling, or that the period doubling is standard but with unusual features.

For the latter interpretation, the stability of the cycles must undergo much more abrupt changes than those of the logistic map. An infinitesimal increase in r causes a superstable 2^k-cycle to period double into a 2^{k+1}-cycle, where $k=1,3,5,7,9,\ldots$. This newborn cycle immediately becomes superstable when r is further increased by an infinitesimal amount. For $k=2,4,6,8,10,\ldots$, the stability of the 2^{k+1}-cycles does not change very abruptly with r, but instead changes very gradually as in the logistic map.

It is difficult to conclude from a numerical study whether the period doublings at the superstable parameters of the linear-logistic map occur by a new mechanism, or by the standard mechanism but with very abrupt changes.

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